

Isotropic systems and the interlace polynomial

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Abstract

Through a series of papers in the 1980's, Bouchet introduced isotropic systems and the Tutte-Martin polynomial of an isotropic system. Then, Arratia, Bollobás, and Sorkin developed the interlace polynomial of a graph in [ABS00] in response to a DNA sequencing application. The interlace polynomial has generated considerable recent attention, with new results including realizing the original interlace polynomial by a closed form generating function expression instead of by the original recursive definition (see Aigner and van der Holst [AvdH04], and Arratia, Bollobás, and Sorkin [ABS04b]). Now, Bouchet [Bou05] recognizes the vertex-nullity interlace polynomial of a graph as the Tutte-Martin polynomial of an associated isotropic system. This suggests that the machinery of isotropic systems may be well-suited to investigating properties of the interlace polynomial. Thus, we present here an alternative proof for the closed form presentation of the vertex-nullity interlace polynomial using the machinery of isotropic systems. This approach both illustrates the intimate connection between the vertex-nullity interlace polynomial and the Tutte-Martin polynomial of an isotropic system and also provides a concrete example of manipulating isotropic systems. We also provide a brief survey of related work.

Key words and phrases: Isotropic systems, interlace polynomial, Tutte-Martin polynomials, circuit partition polynomial, Tutte polynomial, Martin polynomial, graph invariants, circle graphs, intersection graphs, Eulerian graphs, Eulerian circuits, graph polynomials.

Mathematics subject classification: 05C38, 05C45.

1 Introduction

In the 1980's Bouchet unified essential properties of 4-regular graphs and pairs of dual binary matroids through a new algebraic construct, isotropic systems. With the Tutte-Martin polynomial of an isotropic system, he then significantly extended a fundamental relationship between the Tutte polynomial of a planar graph and the Martin polynomial of its medial graph. Then, at the turn of the millennium, Arratia, Bollobas, and Sorkin developed the interlace polynomial of a graph to analyze the interlaced repeated subsequences of nucleotides that can complicate DNA sequencing. Remarkably, despite the very different terminologies, motivations and approaches, the original interlace polynomial of a graph may be realized as the Tutte-Martin polynomial of an associated isotropic system (see Bouchet, [Bou05]).

The intimate connection between the Tutte-Martin polynomial of an isotropic system and the interlace polynomial of a graph suggests that the machinery of isotropic systems may be well-suited to investigating properties of the interlace polynomial, properties of particular current interest due to the interlace polynomial's important applications in the biological sciences and general string reconstruction problems. In the current paper we present an alternative proof for the closed form presentation of the original vertex-nullity interlace polynomial by interpreting it as the Tutte-Martin polynomial of a carefully chosen graphical isotropic system. Our approach places the emphasis on manipulating the Tutte-Martin polynomial of an isotropic system, using it more concretely than typical in prior work, and thus providing an explicit and detailed example of how the machinery of isotropic systems may be used to extract information about the interlace polynomial. En route, we survey related work, providing a consolidated introduction to the interlace and Tutte-Martin polynomials and their interconnections.

A series of papers throughout the 1980's and 1990's, including work by Bouchet [Bou87a], [Bou87b], [Bou88], [Bou89], [Bou91], [Bou93], as well as Bouchet and Ghier [BG96], and Jackson [Jac91], developed the theory of isotropic systems and of Tutte-Martin polynomials of isotropic systems, which are now emerging as valuable tools for extending matroidal and graph theoretical results. In addition to the above-mentioned works, see also in particular Aigner [Aig00], Aigner and Mielke [AM00], Aigner and van der Holst [AvdH04], Allys [All94], as well as Bouchet's series on multimatroids

[Bou97], [Bou98], [Bou01], [Bou98].

In [ABS00], Arratia, Bollobás, and Sorkin defined a one-variable graph polynomial $q_N(G)$ (denoted q there, but we follow later work [ABS04b], reserving q for the two-variable generalization) motivated by questions arising from DNA sequencing by hybridization in Arratia, Bollobás, Coppersmith, and Sorkin [ABCS00]. This polynomial models the interlaced repeated subsequences of DNA that can interfere with the unique reconstruction of the original DNA strand. This work promptly generated further interest and other applications, for example in Aigner, and van der Holst [AvdH04]; Arratia, Bollobás, and Sorkin [ABS04a], [ABS04b]; Balister, Bollobás, Cutler, and Peabody [BBCP02]; Balister, Bollobás, Riordan, and Scott [BBRS01]; Parker and Riera [RP]; and [E-MS]. In [ABS04b], Arratia, Bollobás, and Sorkin define a much richer two-variable interlace polynomial, and show that the original polynomial of [ABS00] and [ABS04a] is a specialization of it, renaming the original interlace polynomial the vertex-nullity interlace polynomial due to its relationship with the two-variable generalization.

The vertex-nullity interlace polynomial was originally defined recursively via a ‘pivot’ operation, and considerable cleverness was required to show that it was in fact well-defined, that is, independent of the order of the pivot operations (in [ABS00] and [ABS04a], Arratia, Bollobás, and Sorkin describe using a computer search to ferret out the necessary identities). One of the most significant recent results, derived from properties of the two-variable interlace polynomial by Arratia, Bollobás, and Sorkin in [ABS04b] and shown with a slightly different formulation by Aigner and van der Holst in [AvdH04], gives the original vertex-nullity interlace polynomial as a closed form generating function expression, and thus an alternative proof that it is well-defined. Both [ABS04b] and [AvdH04] use linear algebra techniques, specifically interpreting the pivot operation in terms of its impact on adjacency matrices. Here, we derive the same result using purely the properties of isotropic systems and the Tutte-Martin polynomial of isotropic systems.

Bouchet [Bou05] has recently explicitly described how the recursive form of the vertex-nullity interlace polynomial of a graph can be derived from the restricted Tutte-Martin polynomial of an isotropic system, a connection implicit by combining previous work by Bouchet [Bou87a], [Bou91], [Bou88]. We exploit this relationship and the machinery of isotropic systems by viewing the interlace polynomial as the Tutte-Martin polynomial of an isotropic

system with a particular graphic presentation. With this, we give an alternative proof for the closed form presentation of the vertex-nullity interlace polynomial. Although our result, that $q(G; x) = \sum_{W \subseteq V(G)} (x - 1)^{|W| - r(A(W))}$, is precisely the formulation given by Arratia, Bollobás, and Sorkin in [ABS04b] rather than that by Aigner and van der Holst in [AvdH04], where the sum involves admissible column sets in an extended adjacency matrix, the latter's approach is somewhat closer, although the reverse, of that presented here.

In [AvdH04], Aigner and van der Holst use linear algebra techniques to obtain a closed form expression for the vertex-nullity interlace polynomial and then compare it to a closed form presentation of the Tutte-Martin polynomial of an isotropic system to conclude that vertex-nullity interlace polynomial and Tutte-Martin polynomial coincide. Here, we reverse this approach, using the main result of Bouchet [Bou05] to first write $q_N(G; x)$ as the Tutte-Martin polynomial of an isotropic system. Then, by choosing the particular isotropic system carefully, we are able to conclude the result of Arratia, Bollobás, and Sorkin [ABS04b], that $q(G; x) = \sum_{W \subseteq V(G)} (x - 1)^{|W| - r(A(W))}$.

2 A brief overview

The vertex-nullity interlace polynomial of a graph was defined recursively by Arratia, Bollobás, and Sorkin in [ABS00] via a pivoting operation and was seen by them in [ABS04b] to be a specialization of a two-variable interlace polynomial, $q(G; x, y)$, with a similar pivot recursion. Let vw be an edge of a graph G , and let A_v , A_w and A_{vw} be the sets of vertices of G adjacent to v only, w only, and to both v and w , respectively. The pivot operation “toggles” the edges between A_v , A_w and A_{vw} , by deleting existing edges and inserting edges between previously non-adjacent vertices. The result of this operation is denoted G^{vw} . More formally, G^{vw} has the same vertex set as G , and edge set equal to the symmetric difference $E(G) \Delta S$, where S is the complete tripartite graph with vertex classes A_v , A_w and A_{vw} . See Figure 1.

[Insert Figure 1: The pivot operation.]

Definition 2.1. *The vertex-nullity interlace polynomial is defined recursively as:*

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$$q_N(G; x) = \begin{cases} x^n & \text{if } G = E_n, \text{ the edgeless graph on } n \text{ vertices,} \\ q_N(G - v; x) + q_N(G^{vw} - w; x) & \text{if } vw \in E(G). \end{cases}$$

This polynomial was shown to be well-defined on all simple graphs by Arratia, Bollobás, and Sorkin in [ABS00].

Definition 2.2. (Arratia, Bollobás, and Sorkin [ABS04b]) *The two-variable interlace polynomial is defined, for a graph G of order n , by*

$$q(G; x, y) = \sum_{S \subseteq V(G)} (x - 1)^{r(G[W])} (y - 1)^{n(G[W])}, \quad (1)$$

where $r(G[W])$ and $n(G[W]) = |W| - r(G[W])$ are, respectively, the $GF(2)$ -rank and nullity of the adjacency matrix of $G[W]$, the subgraph of G induced by W .

Equivalently, the two-variable interlace polynomial can be defined by the following reduction formulas from Arratia, Bollobás, and Sorkin [ABS04b].

For a graph G , for any edge ab where neither a nor b has a loop,

$$q(G) = q(G - a) + q(G^{ab} - b) + ((x - 1)^2 - 1)q(G^{ab} - a - b), \quad (2)$$

for any looped vertex a ,

$$q(G) = q(G - a) + (x - 1)q(G^a - a),$$

and, for the edgeless graph E_n on $n \geq 0$ vertices, $q(E_n) = y^n$. Here G^a is the *local complementation* of G , and is defined as follows. Let $N(a)$ be the neighbors of a , that is, the set $\{w \in V : a \text{ and } w \text{ are joined by an edge}\}$. Thus $a \in N(a)$ iff a is a loop. The graph G^a is equal to G except that $G^a[N(a)] = \overline{G[N(a)]}$, i.e. we “toggle” the edges among the neighbors of a , switching edges to non-edges and vice-versa.

Arratia, Bollobás, and Sorkin show in [ABS04b] that the vertex-nullity interlace polynomial is a specialization of the two-variable interlace polynomial as follows:

$$q_N(G; y) = q(G; 2, y) = \sum_{W \subseteq V(G)} (y - 1)^{n(G[W])}. \quad (3)$$

An equivalent formulation for $q_N(G; x)$ is given by Aigner and van der Holst in [AvdH04]: $q_N(G; g) = \sum_S (y-1)^{co(L_S)}$, where the sum is over certain column sets of L , an extended adjacency matrix of G .

In [ABS00] and [ABS04b], the Arratia, Bollobás, and Sorkin give an interpretation of the vertex-nullity interlace polynomial of a circle graph in terms of the circuit partition, or Martin, polynomial of a related 4-regular Eulerian digraph. A *circle graph* on n vertices is a graph G derived from a chord diagram, where two copies of each of the symbols 1 through n are arranged on the perimeter of a circle, and a chord is drawn between like symbols. Two vertices v and w in G share an edge if and only if their corresponding chords intersect in the chord diagram. See Figure 2.

Circle graphs have also been called alternance graphs by Bouchet in [Bou88] and interlace graphs by Arratia, Bollobás, and Sorkin in [ABS00]. Research on circle graphs includes a complete characterization and a polynomial time algorithm for identifying them. For example, see Bouchet [Bou85], [Bou87b], [Bou87c], [Bou94]; Czemerinski, Duran, Gravano, and Bouchet [CDG02]; Durán [Dur03]; de Fraysseix [Fra84]; Gasse [Gas97]; Read and Rosenstiehl [RR78a], [RR78b]; and Wessel and Pöschel [WP84].

[Insert Figure 2: The circle graph of a chord diagram.]

A *4-regular Eulerian digraph* is a 4-regular directed graph such that, at each vertex, two edges are oriented inward, and two are oriented outward. A 4-regular Eulerian digraph is called a 2-in, 2-out graph in Arratia, Bollobás, and Sorkin [ABS00]. Note that if C is an Eulerian circuit of a 4-regular Eulerian digraph, and we write the vertices along the perimeter of a circle in the order that they are visited by C (each is visited exactly 2 times), and then draw a chord between like vertices, the result is a chord diagram.

Definition 2.3. A *graph state* of a 4-regular Eulerian digraph \vec{G} is the result of replacing each 4-valent vertex v of \vec{G} with two 2-valent vertices each joining an incoming and an outgoing edge originally adjacent to v . Thus a graph state is a disjoint union of consistently oriented cycles. See Figure 3.

Note that graph states (see [E-M98]) are equivalent to the circuit partitions of Arratia, Bollobás, and Sorkin [ABS00] and Bollobás [Bol02], the Eulerian decompositions of Bouchet [Bou88], and the Eulerian k -partitions

of Martin [Mar77] and Las Vergnas [Las83].

[Insert Figure 3: A graph state.]

Definition 2.4. The *circuit partition polynomial* of a 4-regular Eulerian digraph \vec{G} is $f(\vec{G}; x) = \sum_{k \geq 0} f_k(\vec{G}) x^k$, where $f_k(\vec{G})$ is the number of graph states of \vec{G} with k components, defining $f_0(\vec{G})$ to be 1 if \vec{G} has no edges, and 0 otherwise.

The circuit partition polynomial is a simple translation of the Martin polynomial $m(\vec{G}; x)$, defined recursively for 4-regular digraphs by Martin in his 1977 thesis [Mar77], with $f(\vec{G}; x) = xm(\vec{G}; x + 1)$.

Martin also showed that if G is a planar graph and \vec{G}_m is an appropriately oriented medial graph of G , then $m(\vec{G}_m; x) = T(G; x, x)$, where $T(G; x, y)$ is the Tutte polynomial, giving a catalyst for the subsequent development of the Tutte-Martin polynomial.

Las Vergnas found closed forms for the Martin polynomials (for both graphs and digraphs). He also extended their properties to general Eulerian digraphs and further developed their theory (see [Las79], [Las88], [Las83]). The transforms of the Martin polynomials, for arbitrary Eulerian graphs and digraphs, were given in [E-M98], and then aptly named circuit partition polynomials by Bollobás in [Bol02], with splitting identities provided in [Bol02] and [E-M04]. The circuit partition polynomial is also a specialization of a much broader multivariable polynomial, the generalized transition polynomial of [E-MS02], which assimilates such graph invariants as the Penrose polynomial that are not evaluations of the Tutte polynomial.

For circle graphs, the vertex-nullity interlace polynomial and the circuit partition polynomial are related by the following theorem.

Theorem 2.5. (*Arratia, Bollobás, and Sorkin [ABS00], Theorem 6.1*).

If \vec{G} is a 4-regular Eulerian digraph, C is any Eulerian circuit of \vec{G} , and H is the circle graph of the chord diagram determined by C , then $f(\vec{G}; x) = xq_N(H; x + 1)$.

We note that this result is also indicated (in terms of the original Martin polynomial not the circuit partition polynomial) using isotropic systems in

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the work of Bouchet, although the components appear in three separate papers. The necessary results for expressing the Martin polynomial of a graph as a Tutte-Martin polynomial of an isotropic system are given in [Bou87a], and an outline for assembling them to recover the Martin polynomial of a 4-regular Eulerian (di)graph from the appropriate isotropic system appears in [Bou91]. Then, in [Bou88], it is proved that every graphic isotropic system (i.e. those associated in a special way to 4-regular graphs) has a circle graph as a fundamental graph (see [Bou88], theorem 6.3 in particular). Thus the Tutte-Martin polynomial of the graphic isotropic system that is equal to the Martin polynomial of a 4-regular Eulerian digraph \vec{G} is in turn equal to the Tutte-Martin polynomial of the isotropic system of the circle graph of an Eulerian circuit of \vec{G} .

The following brief introduction to isotropic systems follows closely that provided by Bouchet in [Bou05]. Further details can be found for example in [Bou87a] and [Bou91].

Let $K = \{0, x, y, z\}$ be the Klein group, so in particular K is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ with $0 \leftrightarrow (0, 0)$, $x \leftrightarrow (1, 0)$, $y \leftrightarrow (0, 1)$, and $z \leftrightarrow (1, 1)$. We view K as a vector space over $GF(2)$, the two element field $\{0, 1\}$.

There is a bilinear form $\langle \cdot, \cdot \rangle$ uniquely defined on K , mapping K into $GF(2)$ with $\langle a, b \rangle = 1$ if $a \neq b$ and neither a nor b is zero, and $\langle a, b \rangle = 0$ otherwise. For every finite set V , the set K^V of maps from V to K is a vector space of dimension $|V|$ over $GF(2)$. If the elements of V are ordered, then these maps may be thought of as vectors of length $|V|$ where the i^{th} entry is value in K assigned to the i^{th} element of V .

We equip K^V with a bilinear form $\langle \cdot, \cdot \rangle$ defined by $\langle A, B \rangle = \sum_{v \in V} \langle A(v), B(v) \rangle$ for all A, B in K^V .

We set $K' = K \setminus \{0\} = \{x, y, z\}$.

Definition 2.6. *An isotropic system is a pair $S = (L, V)$, where V is a finite set and L is a vector subspace of K^V such that $\dim L = |V|$ and $\langle A, B \rangle = 0$, for all $A, B \in L$.*

Let G be a simple graph with vertex set V and the edge set E . The set $\mathcal{P}(V)$ of subsets of V is considered as a vector space over $GF(2)$. Thus, if V is ordered, a subset of V may be identified with a vector of length $|V|$, with i^{th} entry 1 if the i^{th} element of V is in the subset and 0 else. Addition of two

vectors in the vector space thus corresponds to the symmetric difference of the underlying subsets.

For $v \in V$, let $N(v)$ be the neighbors of v , that is, the set $\{w \in V : v \text{ and } w \text{ are joined by an edge}\}$. We set $N(P) = \sum_{v \in P} N(v)$, for $P \subseteq V$. Note that $N(P)$, since we are summing in $GF(2)$, is simply the set of vertices of G that are adjacent to an odd number of the vertices of P . For $X \in K^V$ and $P \subseteq V$, the vector $Y \in K^V$ defined by $Y(v) = X(v)$ if $v \in P$ and $Y(v) = 0$ if $v \notin P$, will be denoted by $X(P)$.

Theorem 2.7. (*Bouchet [Bou88] Theorem 3.1*) *Let G be a simple graph with vertex set V . Let $A, B \in K^V$ be such that $A(v) \neq B(v)$, for all $v \in V$, and set $L = \{A(P) + B(N(P)) : P \subseteq V\}$. Then $S = (L, V)$ is an isotropic system.*

The triple (G, A, B) is called a *graphic presentation* of S .

For $X \in K^V$ we set $\hat{X} = \{Y \in K^V : Y(v) \in \{0, X(v)\}, v \in V\}$. Equivalently, $\hat{X} = \{X(P) \mid P \subseteq V\}$. We note that \hat{X} is a vector subspace of K^V with $X(P) + X(Q) = X(P \Delta Q)$.

Definition 2.8. *For $C \in K^V$, the restricted Tutte-Martin polynomial $m(S, C; \xi)$ is defined by*

$$m(S, C; \xi) = \sum (\xi - 1)^{\dim(L \cap \hat{F})},$$

where the sum is over $F \in K^V$ such that $F(v) \neq C(v)$, for all $v \in V$.

Definition 2.8 gives the Tutte-Martin polynomial of a general isotropic system. However, in the special case that the isotropic system is given by a graphic presentation, we have the following central result.

Theorem 2.9. (*Bouchet [Bou05] Equality (5)*) *If G is a simple graph and S is the isotropic system defined by a graphic presentation (G, A, B) , then*

$$q_N(G; \xi) = m(S, A + B; \xi).$$

At the heart of the proof is that $q_N(G; \xi)$ may be recursively defined by the following relations:

- (1) $q_N(G; \xi) = 1$ if G is the empty graph.

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- (2) $q_N(G; \xi) = \xi q_N(G \setminus v; \xi)$ if v is an isolated vertex,
- (3) $q_N(G; \xi) = q_N(G \setminus v; \xi) + q_N(G * v w v \setminus v; \xi)$ if $vw \in E$.

Here $G * v$ is the local complementation of G at v , which replaces the subgraph induced by $N(v)$ by the complementary subgraph, precisely as defined above, but the notation of Bouchet [Bou05] differs from Arratia, Bollobás, and Sorkin [ABS04b]. $G * v w v$ is the iterated operation $G * v * w * v$. See Figure 4, and compare to Figure 1.

[Insert Figure 4: complementation along an edge].

Aigner and van der Holst give an alternative proof of this same result in [AvdH04]. If G is a simple graph on n vertices, then let L be the $(n \times 2n)$ matrix over $GF(2)$ whose first n columns are the adjacency matrix of G , and whose last n columns are the $n \times n$ identity matrix, labeling the rows of L by $1, \dots, n$, and the columns by $1, \dots, n, \bar{1}, \dots, \bar{n}$. A column set S is *admissible* if $|S \cap \{i, \bar{i}\}| = 1$ for all i , and L_S is the $(n \times n)$ submatrix with column set S . Having first used a linear algebra approach to show that $q_N(G; x) = \sum_S (x-1)^{co(L_S)}$, where the sum is over all admissible column sets S , and $co(L_S)$ is the corank, Aigner and van der Holst then compare this to the formula of Definition 2.8 to conclude Theorem 2.9.

3 Interpretation of $q_N(G)$ via Isotropic systems

That $q_N(G; x) = m(S, C; x)$ gives an interpretation of what $q_N(G; x)$ counts, but only with respect to the underlying isotropic system. Here, by translating the isotropic machinery into graph theoretical terms, we give an alternative proof of the generating function formulation of Arratia, Bollobás, and Sorkin [ABS04b], namely that

$$q_N(G; x) = \sum_{W \subseteq V(G)} (x-1)^{|W| - r(M(W))}, \quad (4)$$

where $M(W)$ is the adjacency matrix of G restricted to W and r is the $GF(2)$ rank function.

Given a simple graph G with vertex set V , we choose an isotropic system with a particular graphic presentation as follows. For $p \in \{x, y, z\}$ we define $\bar{p} : V \rightarrow K$ as the element of K^V such that $\bar{p}(v) = p$ for all $v \in V$. Let $A = \bar{x}$, $B = \bar{y}$, and $C = A + B = \bar{z}$. Now let S be the isotropic system whose graphic presentation is (G, A, B) as given by Theorem 2.7.

Using Definition 2.8 and recalling that $\widehat{F} = \{F' \in K^V \mid F'(v) = F(v) \text{ or } F'(v) = 0\}$, we have, for $A = \bar{x}$, $B = \bar{y}$, that

$$m(S, \bar{z}; \xi) = \sum (\xi - 1)^{\dim(L \cap \widehat{W})},$$

where the sum is over $F \in \{x, y\}^V$, and where $L = \{A(P) + B(N(P)) : P \subseteq V\}$.

Write L_P for $A(P) + B(N(P))$, so,

$$L_P(v) = \begin{cases} x & \text{if } v \in P, v \notin N(P), \\ y & \text{if } v \notin P, v \in N(P), \\ z & \text{if } v \in P \cap N(P), \\ 0 & \text{else.} \end{cases} \quad (5)$$

If $L_P \in L \cap \widehat{F}$ and $F \in \{x, y\}^V$, then $L_P \in \{0, x, y\}^V$. For $P \subseteq V$, we denote by $G \mid P$ the subgraph of G induced by P , and we write F_x for $F^{-1}(x)$ and F_y for $F^{-1}(y)$. An *even* graph has all vertices of even degree and is not necessarily connected.

Lemma 3.1. *For $P \subseteq V$ the following statements are equivalent:*

1. $L_P \in \{0, x, y\}^V$,
2. $G \mid P$ is an even subgraph of G .

Proof. It follows from (5) that $L_P \in \{0, x, y\}^V$ if and only if $P \cap N(P) = \emptyset$. Since $N(P)$ is the set of vertices adjacent to an odd number of vertices of P , this is equivalent to saying that every $v \in P$ has an even number of neighbors in P . That is, if and only if $G \mid P$ is an even subgraph of G . \square

Since Aigner and van der Holst [AvdH04] and the current work, although giving reverse approaches, are in fact examining the same phenomenon, it is

not surprising that they should at some point converge, and we note that the preceding Lemma 3.1 is essentially Lemma 3 of [AvdH04].

Lemma 3.2. *Let $F \in \{x, y\}^V$ and $P \subseteq V$. Then the following statements are equivalent:*

1. $L_P \in \widehat{F}$,
2. $G \mid P$ is an even subgraph of $G \mid F_x$ and $N(P) \subseteq F_y$.

Proof. By the definition of L_P we have that $L_P \in \widehat{F}$ if and only if there is and $R \subseteq V$ such that

$$L_P(v) = F(R) = \begin{cases} x & \text{if } v \in R \cap F_x \\ y & \text{if } v \in R \cap F_y \\ 0 & \text{if } v \notin R \end{cases} \quad (6)$$

Comparing (6) to (5), we note that $L_P = F(R)$ if and only if $P = R \cap F_x$, $N(P) = R \cap F_y$ and $R = P \cup N(P)$. That is, $L_P = F(R)$ if and only if $P \subseteq F_x$, $N(P) \subseteq F_y$, $P \cap N(P) = \emptyset$ and $R = P \cup N(P)$.

By Lemma 3.1, $P \cap N(P) = \emptyset$ and $F \subseteq F_x$ if and only if $G \mid P$ is an even subgraph of $G \mid F_x$.

The desired conclusion follows readily. □

By Lemma 3.2, we have that

$$L \cap \widehat{F} = \{L_P \in L : G \mid P \text{ is an even subgraph of } G \mid F_x \text{ and } N(P) \subseteq F_y\}.$$

Next we determine, in terms of the graph G , the dimension of $L \cap \widehat{F}$ as a subspace of K^V .

Write $star(v)$ for the subgraph of edges incident to v . Note that $N(P) \subseteq F_y = V \setminus F_x$ if and only if none of the elements of $F_x \setminus P$ is in $N(P)$. That is, $N(P) \subseteq F_y$ if and only if for all $v \in F_x \setminus P$ the cardinality of $star(v) \cap P$ is even. On the other hand, $G \mid P$ is an even subgraph of $G \mid F_x$ if and only if $P \subseteq F_x$ and for all $v \in P$ the cardinality of $star(v) \cap P$ is even. Therefore, by Lemma 3.2,

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$L \cap \widehat{F} = \{L_P : P \subseteq F_x \text{ and for all } v \in F, \text{ the cardinality of } \text{star}(v) \cap P \text{ is even}\}.$

Recall that the power set $\mathcal{P}(V)$ is a vector space over $GF(2)$ where addition is the symmetric differences of subsets of V . Let U be the subspace of $\mathcal{P}(V)$ defined by $U = \{P \subseteq F_x : \text{for all } v \in F, \text{ the cardinality of } \text{star}(v) \cap P \text{ is even}\}.$

Lemma 3.3. *Let F be an element of $\{x, y\}^V$. Then $\dim(L \cap \widehat{F}) = \dim(U)$.*

Proof. Let $\Psi : L \cap \widehat{F} \rightarrow U$ be the function given by $\Psi(L_P) = P$.

It is straightforward to prove that Ψ is a well defined bijection.

We will prove that Ψ is also a homomorphism. Let P, P' be arbitrarily given subsets of F_x . As in Lemma 3.2, $L_P = F(P \cup N(P))$, where the union is disjoint. Note that, by the definition of the neighborhood of a set, $N(P) + N(P') = \sum_{v \in P} N(v) + \sum_{v \in P'} N(v) = \sum_{v \in P+P'} N(v) = N(P+P')$.

Since $L \cap \widehat{F}$ is a subspace of \widehat{F} , and we noted below Theorem 2.7 that $F(R) + F(Q) = F(R \Delta Q) = F(R + Q)$, we have the following:

$$\begin{aligned} L_P + L_{P'} &= F(N \cup N(P)) + F(P' \cup N(P)) \\ &= F(P \cup N(P)) \Delta F(P' \cup N(P')) \\ &= F(P \Delta P') \cup (N(P) \Delta N(P')) \\ &= F((P \Delta P') \cup N(P \Delta P')) \\ &= L_{P \Delta P'} = L_{P+P'}. \end{aligned}$$

Thus Ψ is a homomorphism.

It follows that $\dim(L \cap \widehat{F}) = \dim(U)$. □

Lemma 3.4. *Let F be an element of $\{x, y\}^V$. Then*

$$\dim(L \cap \widehat{F}) = \dim(U) = |F_x| - r(M),$$

where M is the adjacency matrix of $G \mid F_x$.

Proof. Let $\psi : \mathcal{P}(F_x) \rightarrow GF(2)^{F_x}$ be the homomorphism whose matrix M is the adjacency matrix of $G \mid F_x$. That is, the rows and columns of M are indexed by the elements of $G \mid F_x$ and $M_{ij} = 1$ iff i and j are joined by an edge in $G \mid F_x$. Also $\psi(P) = M \cdot \chi_P$, where χ_P is the vector identified with P written as a column.

We have that $P \in \text{Ker}(\psi)$ if and only if $v(P) = M \cdot \chi_P = 0$ (the all zero column). That is, for every $i \in F_x$, $0 = \sum_{j \in F_x} M_{ij} \chi_{P_j} = \sum_{j \in P} M_{ij}$. In other words, for every $i \in F_x$, the cardinality of $\text{star}(v) \cap P$ is even.

Thus $L \cap \widehat{F}$ is the kernel of ψ and $\dim(\text{Im}(\psi)) = r(M)$, where r is the $GF(2)$ rank function. On the other hand $\dim(\mathcal{P}(F_x)) = |F_x|$.

Therefore

$$\dim(L \cap \widehat{F}) = \dim(U) = |F_x| - r(M).$$

□

Theorem 3.5. *Let G be a simple graph. Let $q_N(G; x)$ be the vertex-nullity interlace polynomial of G . Then*

$$q_N(G; x) = \sum_{W \subseteq V(G)} (x - 1)^{|W| - r(M(W))},$$

where $M(W)$ is the adjacency matrix of $G \mid W$.

Proof. By Theorem 2.9, given a simple graph G on the vertex set V , supplementary vectors A and B of K^V , and $L = \{A(N(P)) + B(P) : P \in \mathcal{P}(V)\}$, then the pair $S = (L, V)$ is an isotropic system. Moreover, the mapping $X \rightarrow A(N(P)) + B(P)$ is a linear bijection from $\mathcal{P}(V)$ onto L . Let $A = \bar{x}$, $B = \bar{y}$ and $C = A + B = \bar{z}$ as before. It follows that there is an isotropic system S such that (G, A, B) is its graphic presentation.

From Definition 2.8 and Theorem 2.9,

$$q_N(G; \xi) = m(S, C; \xi) = \sum (\xi - 1)^{\dim(L \cap \widehat{F})},$$

where the sum is over $F \in \{x, y, x\}^V$ such that $F(v) \neq C(v)$ for all $v \in V$.

Therefore, for $A = \bar{x}$, $B = \bar{y}$ and $C = A + B = \bar{z}$, we get

$$m(S, C; \xi) = \sum (\xi - 1)^{\dim(L \cap \widehat{F})},$$

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where the sum is over $F \in \{x, y\}^V$.

By Lemma 3.4, $\dim(L \cap \widehat{F}) = \dim(U) = |F_x| - r(M(F_x))$, where $M(F_x)$ is the adjacency matrix of $G \upharpoonright F_x$ and $F_x = F^{-1}(x)$. Thus,

$$m(S, C; \xi) = \sum_{F_x \subseteq V} (\xi - 1)^{|F_x| - r(M(F_x))}.$$

Therefore,

$$q_N(G; x) = \sum_{W \subseteq V(G)} (x - 1)^{|W| - r(M(W))},$$

where $M(W)$ is the adjacency matrix of $G \upharpoonright W$.

□

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Figure 1: Pivoting on the edge vw .

A_v , A_w and A_{vw} are the sets of vertices of G adjacent to v only, w only, and to both v and w , respectively. These sets are constant in all the diagrams. Vertices of G adjacent to neither v nor w are omitted. Heavy lines indicate that all edges are present, and dotted lines represent non-edges.

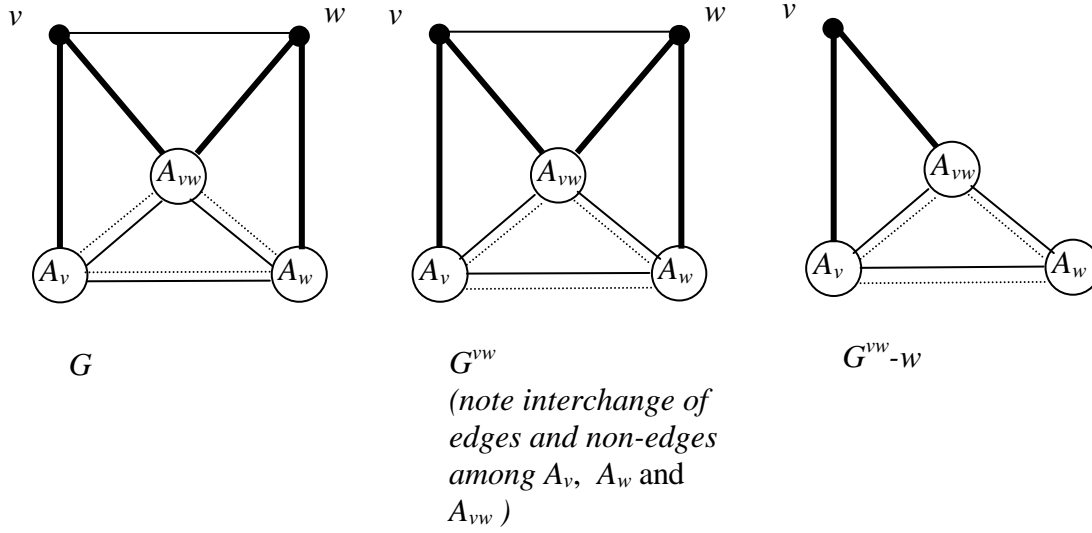
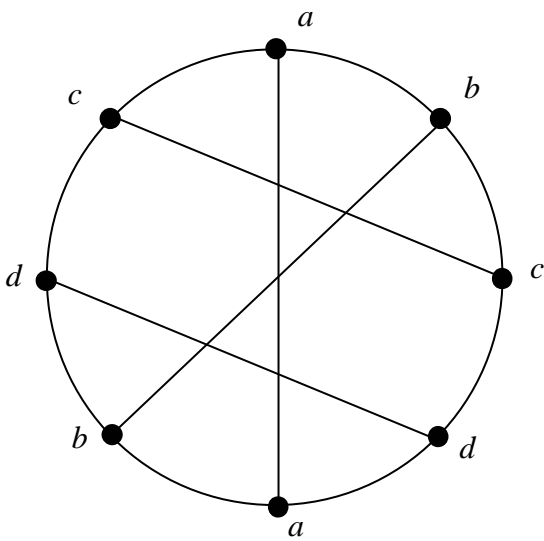
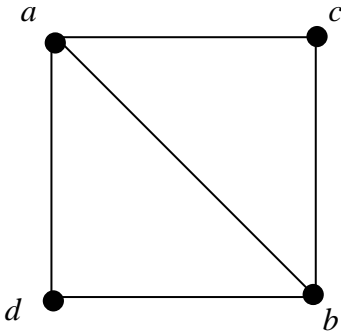


Figure 2: A chord diagram and its associated circle graph.

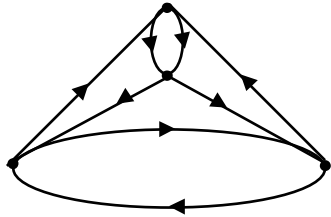


A chord diagram

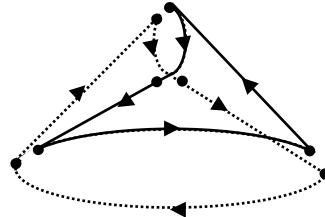


The associated circle graph

Figure 3: A graph state.

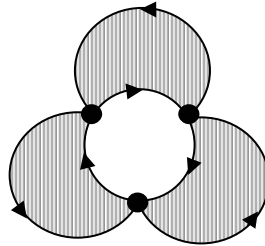
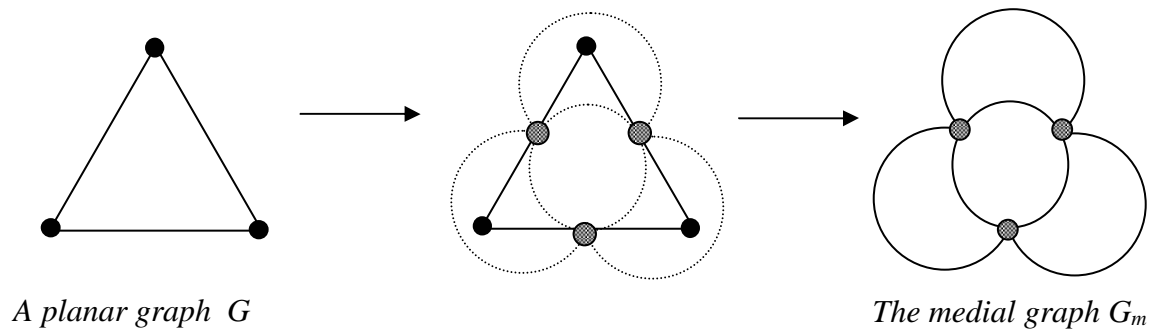


A 4-regular Eulerian digraph \vec{G}



A graph state of \vec{G} with 2 components

Figure 4: The oriented medial graph.



G_m with the vertex faces colored black,
oriented so that black faces are to the left of
each edge.